A GENERALIZATION OF EHRENFEUCHT'S GAME AND SOME APPLICATIONS

BY

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ABSTRACT

The concept of α, β-elementary equivalence of two relational structures is defined. Sufficient and necessary conditions are given for this notion by generalizing Ehrenfeucht's game in algebraic terms. Some results, in a first order language with generalized quantifier, are obtained.

In this paper we generalize a method by Ehrenfeucht [1]. (The main results of this generalization were announced first in [9].) A similar generalization was made independently, at about the same time by Lipner [5]. From the method suggested here, one can obtain immediately Lipner's method as a special case. To be more specific, let $L(Q)$ be the language obtained from a countable ordinary first order language L by adding the quantifier Q (L includes equality). The α-satisfaction for $L(Q)$ is obtained by interpreting Q as: "there exist at least α elements" (α, β, γ will denote here infinite cardinals). We use $\models_\alpha$ to denote α-satisfaction. Let $\mathfrak{A}$, $\mathfrak{B}$ be two models for L. We write $\mathfrak{A}^\alpha \equiv {}^\beta\mathfrak{B}$ if for every sentence ϕ in $L(Q)$, $\mathfrak{A} \models_\alpha \phi$ iff $\mathfrak{B} \models_\beta \phi$. In this paper we give a criterion for $\mathfrak{A}$, $\mathfrak{B}$ to fulfill $\mathfrak{A}^\alpha \equiv {}^\beta\mathfrak{B}$ and we mention a few applications of this criterion.

It is possible to formulate the criterion in game theoretic terms and in algebraic terms. Although it seems that the former are more intuitive, we prefer the latter because in many cases (especially the more complicated ones) it is much simpler to formulate proofs in algebraic terms. The reader, who is familiar with

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* This paper is part of the author's doctoral dissertation which was prepared under the supervision of Professor Azriel Levy at the Hebrew University of Jerusalem. The author wishes to thank Professor Levy for his kind assistance.

Received June 22, 1972.

Ehrenfeucht's game [1], may wish to find for himself equivalent game theoretic concepts or refer to Lipner [5] and Slomson [8].

1. The concept of α, β, n-local isomorphism

We suppose that L does not contain function symbols. Let $\mathfrak{A}, \mathfrak{B}$ be two models for L, $|\mathfrak{A}| \leqq \alpha$, $|\mathfrak{B}| \leqq \beta$. Note that by Fuhrken [2], for every $\mathfrak{A}_1$ such that $|\mathfrak{A}_1| > \alpha$, there exists $\mathfrak{A}$ such that $|\mathfrak{A}| = \alpha$ and $\mathfrak{A}_1^{\alpha} \equiv {}^{\alpha}\mathfrak{A}$. Suppose also that the domains of $\mathfrak{A}$ and $\mathfrak{B}$ are disjoint, and $\alpha \geqq \beta$. By a_i, a denote always elements of $\mathfrak{A}$, and by b_i, b denote always elements of $\mathfrak{B}$. The letters i–p will denote positive integers.

DEFINITION 1.1. Suppose that the correspondence: $a_i \leftrightarrow b_i$, $i = 1, \cdots, k$, is an isomorphism. We define by induction on $n, n \geqq k$, when a set of sequences is called an α, β, n-extension of $\langle a_i, b_i \rangle$, $i = 1, \cdots, k$.

An α, β, k-extension of $\langle a_i, b_i \rangle$, $i = 1, \cdots, k$ is the set whose only element is the sequence $\langle a_i, b_i \rangle$, $i = 1, \cdots, k$.

Suppose that we have defined when a set of sequences is an α, β, n-extension of $\langle a_i, b_i \rangle$, $i = 1, \cdots, k$. Let Γ be a set of sequences of length $n+1$. It is called an $\alpha, \beta, n+1$-extension of $\langle a_i, b_i \rangle$, $i = 1, \cdots, k$ if the following conditions hold:

(a) The first k pairs in each sequence in Γ are the pairs $\langle a_i, b_i \rangle$, $i = 1, \cdots, k$ above (only the $n+1-k$ last pairs in each sequence are possibly new pairs). There exists Γ', an α, β, n-extension of $\langle a_i, b_i \rangle$, $i = 1, \cdots, k$, such that the set of all segments of length n of sequences in Γ is Γ'.

(b) Denote by $\Gamma_{\langle a_i, b_i \rangle}$, $i = 1, \cdots, n$ the set of all sequences in Γ whose segment of length n is $\langle a_i, b_i \rangle$, $i = 1, \cdots, n$. Then for every non-empty $\Gamma_{\langle a_i, b_i \rangle, i=1,\ldots,n}$ there exists a family of functions F that fulfills the following:

(1) Every $f \in F$ is a function from $\mathfrak{A} \cup \mathfrak{B}$ to $\mathfrak{A} \cup \mathfrak{B}$ such that $f(x) \in \mathfrak{B}$ iff $x \in \mathfrak{A}$.

(2) For every set $A' \subseteq \mathfrak{A}$ such that $|A| = \alpha$ there exists $f \in F$ such that $|f(A')| = \beta$.

(3) For every set $B' \subseteq \mathfrak{B}$ such that $|B'| = \beta$ there exist $f \in F$ and a set $A' \subseteq \mathfrak{A}$ such that $|A'| = \alpha$ and $f(A') \subseteq B'$.

(4) For every $f \in F$, $a_{n+1} \in \mathfrak{A}$, $b_{n+1} \in \mathfrak{B}$ such that $b_{n+1} = f(a_{n+1})$ or $a_{n+1} = f(b_{n+1})$, the correspondence: $a_i \leftrightarrow b_i$, $i = 1, \cdots, n+1$ is an isomorphism, and the sequence $\langle a_i, b_i \rangle$, $i = 1, \cdots, n+1$ is in Γ. Γ contains only sequences obtained from $\langle a_i, b_i \rangle$, $i = 1, \cdots, n$ by some $f \in F$.

LEMMA 1.1. *Suppose $\mathfrak{A}, \mathfrak{B}$ are given and the correspondence: $a_i \leftrightarrow b_i$, $i = 1, \cdots, k$ is an isomorphism. Let Γ be an α, β, n-extension of $\langle a_i, b_i \rangle$, $i = 1, \cdots, k$, $n \geqq k$. Suppose also that $k \leqq m \leqq n$. Denote by Γ^m the set of all segments of sequences in Γ which have the length m. Then*

(1) *Γ^m is an α, β, m-extension of $\langle a_i, b_i \rangle$, $i = 1, \cdots, k$.*

(2) *If the sequence $\langle a_i, b_i \rangle$, $i = 1, \cdots, m$ is in Γ^m, then $\Gamma_{\langle a_i, b_i \rangle, i=1,\ldots,m}$ is an α, β, n-extension of $\langle a_i, b_i \rangle$, $i = 1, \cdots, m$.*

PROOF. The result follows by induction on n using Definition 1.1.

DEFINITION 1.2. We say that $\mathfrak{A}, \mathfrak{B}$ are α, β, n-locally isomorphic over $\langle a_i, b_i \rangle$, $i = 1, \cdots, k$, $k \leqq n$, if there exists an α, β, n-extension of $\langle a_i, b_i \rangle$, $i = 1, \cdots, k$.

LEMMA 1.2. *A sufficient and necessary condition for $\mathfrak{A}, \mathfrak{B}$ to be α, β, n-locally isomorphic over $\langle a_i, b_i \rangle$, $i = 1, \cdots, k-1$, $n \geqq k$, is an existence of a family of functions F that fulfills conditions* (b) (1)–(3) *in Definition* 1.1 *such that for every $f \in F$, $a_k \in \mathfrak{A}$, $b_k \in \mathfrak{B}$, if $b_k = f(a_k)$ or $a_k = f(b_k)$ then $\mathfrak{A}, \mathfrak{B}$ are α, β, n-locally isomorphic over $\langle a_i, b_i \rangle$, $i = 1, \cdots, k$.*

PROOF. To show that the condition is sufficient, suppose $b_k = f(a_k)$ or $a_k = f(b_k)$ for certain $f \in F$. Denote by Γ_{a_k, b_k} an α, β, n-extension of $\langle a_i, b_i \rangle$, $i = 1, \cdots, k$. Define $\Gamma = U(\Gamma_{a_k, b_k}:\ a_k \in \mathfrak{A},\ b_k \in \mathfrak{B},\ f \in F$ and $b_k = f(a_k)$ or $a_k = f(b_k))$. It is easy to see that Γ is an α, β, n-extension of $\langle a_i, b_i \rangle$, $i = 1, \cdots, k-1$
To show that the condition is necessary, use Lemma 1.1.

Using the symbol $\breve{Q}$ to denote $\neg Q \neg$ it is known that each formula ψ in $L(Q)$ has a prenex normal form in $L(Q)$, namely, a formula ϕ of the type $G_1 x_1 \cdots G_m x_m \chi$, where G_i, $i = 1, \cdots, m$, is one of the quantifiers $\exists, \forall, Q, \breve{Q}$, χ is quantifier-free and for every model $\mathfrak{D}$ of L, $|\mathfrak{D}| \geqq \alpha$, holds: $\mathfrak{D} \models_\alpha \psi \leftrightarrow \phi$.

NOTATION 1.3. Suppose $0 \leqq k \leqq n$, $n \geqq 1$. By $P_{n,k}$ denote the set of all formulae ψ in $L(Q)$ such that:

(1) ψ is in a prenex normal form in $L(Q)$.

(2) ψ has exactly n different variables.

(3) ψ has exactly k different free variables.

THEOREM 1.3. *If $\mathfrak{A}, \mathfrak{B}$ are α, β, n-locally isomorphic over $\langle a_i, b_i \rangle$, $i = 1, \cdots, k$,*

$0 \leqq k \leqq n$, $n \geqq 1$, *then for every* ψ *in* $P_{n,k}$ *the following holds*: $\mathfrak{A} \vDash_\alpha \psi(a_1, \cdots, a_k)$ *iff* $\mathfrak{B} \vDash_\beta \psi(b_1, \cdots, b_k)$.

PROOF. The theorem follows by induction on $n - k$. When $n - k = 0$, then ψ is quantifier free. Our assumption implies that the correspondence: $a_i \leftrightarrow b_i$, $i = 1, \cdots, n$ is an isomorphism, namely, for every quantifier free formula ψ that has exactly n different variables, we have: $\mathfrak{A} \vDash_\alpha \psi(a_1, \cdots, a_n)$ iff $\mathfrak{B} \vDash_\beta \psi(b_1, \cdots, b_n)$.

Suppose now that the theorem is true for $n - k$, $k \geqq 1$, and prove it for $n - (k - 1)$. Our assumption now is that $\mathfrak{A}, \mathfrak{B}$ are α, β, n-locally isomorphic over $\langle a_i, b_i \rangle$, $i = 1, \cdots, k - 1$. By Lemma 1.2, there exists a family of functions F that fulfills conditions (b) (1)-(3) of Definition 1.1 such that for every $f \in F$, $a_k \in \mathfrak{A}$, $b_k \in \mathfrak{B}$, if $b_k = f(a_k)$ or $a_k = f(b_k)$ then $\mathfrak{A}, \mathfrak{B}$ are α, β, n-locally isomorphic over $\langle a_i, b_i \rangle$, $i = 1, \cdots, k$. Note that in order to prove the theorem, it is sufficient to show that for every ψ in $P_{n,k-1}$, if $\mathfrak{A} \vDash_\alpha \psi(a_1, \cdots, a_{k-1})$ then $\mathfrak{B} \vDash_\beta \psi(b_1, \cdots, b_{k-1})$ (this is true because for every ψ in $P_{n,k-1}$, $\neg \psi$ is also in $P_{n,k-1}$). So let ψ be in $P_{n,k-1}$. It has the form $Gx_k\phi$ where ϕ is in $P_{n,k}$ and G is one of the quantifiers $\exists, \forall, Q, \mathfrak{Q}$.

(1) Suppose $\psi = \exists x_k \phi$ and $\mathfrak{A} \vDash_\alpha \exists x_k \phi(a_1, \cdots, a_{k-1}, x_k)$. There exists $a_k \in \mathfrak{A}$ such that $\mathfrak{A} \vDash_\alpha \phi(a_1, \cdots, a_{k-1}, a_k)$. Let f be any function in F. Denote $b_k = f(a_k)$. $\mathfrak{A}, \mathfrak{B}$ are α, β, n-locally isomorphic over $\langle a_i, b_i \rangle$, $i = 1, \cdots, k$, so, by the induction hypothesis, $\mathfrak{B} \vDash_\beta \phi(b_1, \cdots, b_{k-1}, b_k)$. Therefore, $\mathfrak{B} \vDash_\beta \exists x_k \phi(b_1, \cdots, b_{k-1}, x_k)$.

(2) Suppose $\psi = \forall x_k \phi$ and $\mathfrak{A} \vDash_\alpha \forall x_k \phi(a_1, \cdots, a_{k-1}, x_k)$. If $\mathfrak{B} \vDash_\beta \neg \forall x_k \phi$ $(b_1, \cdots, b_{k-1}, x_k)$, there exists $b_k \in \mathfrak{B}$ such that $\mathfrak{B} \vDash_\beta \neg \phi(b_1, \cdots, b_{k-1}, b_k)$. Let f be any function in F. Denote $a_k = f(b_k)$. Again, $\mathfrak{A}, \mathfrak{B}$ are α, β, n-locally isomorphic over $\langle a_i, b_i \rangle$, $i = 1, \cdots, k$, so, by the induction hypothesis, $\mathfrak{A} \vDash_\alpha \neg \phi(a_1, \cdots, a_{k-1}, a_k)$, a contradiction.

(3) Suppose $\psi = Qx_k\phi$ and $\mathfrak{A} \vDash_\alpha Qx_k\phi(a_1, \cdots, a_{k-1}, x_k)$. Hence, there exists $A' \subseteq \mathfrak{A}$, $|A'| = \alpha$, such that for every $a_k \in A'$, $\mathfrak{A} \vDash_\alpha \phi(a_1, \cdots, a_{k-1}, a_k)$. By (b) (2) of Definition 1.1, there exists $f \in F$ such that $|f(A')| = \beta$. Denote $b_k = f(a_k)$ for every $a_k \in A'$. Since $\mathfrak{A}, \mathfrak{B}$ are α, β, n-locally isomorphic over $\langle a_i, b_i \rangle$, $i = 1, \cdots, k$, we obtain by the induction hypothesis: $\mathfrak{B} \vDash_\beta \phi(b_1, \cdots, b_{k-1}, b_k)$ for every $a_k \in A'$. Therefore $|\{b : \mathfrak{B} \vDash_\beta \phi(b_1, \cdots, b_{k-1}, b)\}| = \beta$ and $\mathfrak{B} \vDash_\beta Qx_k\phi$ $(b_1, \cdots, b_{k-1}, x_k)$.

(4) Suppose $\psi = \mathfrak{Q}x_k\phi$ and $\mathfrak{A} \vDash_\alpha \mathfrak{Q}x_k\phi(a_1, \cdots, a_{k-1}, x_k)$. If $\mathfrak{B} \vDash_\beta \neg \mathfrak{Q}x_k\phi$ $(b_1, \cdots, b_{k-1}, x_k)$ then there exists $B' \subseteq B$, $|B'| = \beta$, such that for every $b \in B'$,

$\mathfrak{B} \models_\beta \neg \phi(b_1, \cdots, b_{k-1}, b)$. By (b) (3) of Definition 1.1 there exist $f \in F$ and $A' \subseteq \mathfrak{A}$ such that $|A'| = \alpha$ and $f(A') \subseteq B'$. Denote again $b_k = f(a_k)$ for every $a_k \in A'$. Then $\mathfrak{B} \models_\beta \neg \phi(b_1, \cdots, b_{k-1}, b_k)$ for every $a_k \in A'$. Since $\mathfrak{A}, \mathfrak{B}$ are α, β, n-locally isomorphic over $\langle a_i, b_i \rangle$, $i = 1, \cdots, k$, for every $a_k \in A'$, we obtain by the induction hypothesis: $\mathfrak{A} \models_\alpha \neg \phi(a_1, \cdots, a_{k-1}, a_k)$ for every $a_k \in A'$. So $\mathfrak{A} \models_\alpha Qx_k \neg \phi(a_1, \cdots, a_{k-1}, x_k)$, namely, $\mathfrak{A} \models_\alpha \neg \breve{Q}x_k \phi(a_1, \cdots, a_{k-1}, x_k)$, a contradiction.

Definition 1.4. $\mathfrak{A}, \mathfrak{B}$ will be called α, β, n-locally isomorphic if $\mathfrak{A}, \mathfrak{B}$ are α, β, n-locally isomorphic over the empty sequence.

We formulate now a lemma which we shall use in following sections to prove that two given structures are α, β, n-locally isomorphic.

Lemma 1.4. *$\mathfrak{A}, \mathfrak{B}$ are $\alpha, \beta, n + 1$-locally isomorphic iff there exists an α, β, n-extension Γ of the empty sequence such that for every sequence $\langle a_i, b_i \rangle$, $i = 1, \cdots, n$, in Γ there exists a family of functions F that fulfills conditions* (b) (1)–(4) *in Definition* 1.1.

Proof. The result follows immediately from the definitions.

Definition 1.5. $\mathfrak{A}, \mathfrak{B}$ will be called α, β, n-elementary equivalent if for every sentence ϕ in $P_{n.0}$ the following holds: $\mathfrak{A} \models_\alpha \phi$ iff $\mathfrak{B} \models_\beta \phi$. If $\mathfrak{A}, \mathfrak{B}$ are α, β, n-elementary equivalent denote: $\mathfrak{A}^\alpha \equiv {}_n^\beta \mathfrak{B}$.

Theorem 1.5. (1) *If $\mathfrak{A}, \mathfrak{B}$ are α, β, n-locally isomorphic then $\mathfrak{A}^\alpha \equiv {}_n^\beta \mathfrak{B}$.*

(2) *If $\mathfrak{A}, \mathfrak{B}$ are α, β, n-locally isomorphic for every n then $\mathfrak{A}^\alpha \equiv {}^\beta \mathfrak{B}$.*

Proof. The conclusion follows from Theorem 1.3 (for $k = 0$) and from the fact that every sentence in $L(Q)$ has an equivalent prenex normal form in $L(Q)$.

From now on suppose that L contains only a finite number of predicates. We shall say briefly that L is finite. For simplicity of formulation, suppose also that L does not contain individual constants.

Lemma 1.6. *Suppose L is finite, $n \geqq 1$, $0 \leqq k \leqq n$. Then there exists a sequence of formulae in $L(Q)$, χ_l^k, $l = 1, \cdots, m_k$, each formula in the sequence having exactly k different free variables, such that for every $\mathfrak{A}, \mathfrak{B}$ (models for L that fulfill: $|\mathfrak{A}| \leqq \alpha$, $|\mathfrak{B}| \leqq \beta$) for every $a_1, \cdots, a_k \in \mathfrak{A}$ and $b_1, \cdots, b_k \in \mathfrak{B}$,* *we have*: $\mathfrak{A},\mathfrak{B}$ *are* α,β,n*-locally isomorphic over* $\langle a_i,b_i\rangle$, $i = 1,\cdots,k$ *iff there exists* j, $1 \leqq j \leqq m_k$, *such that* $\mathfrak{A} \models_\alpha \chi_j^k(a_1,\cdots,a_k)$ *and* $\mathfrak{B} \models_\beta \chi_j^k(b_1,\cdots,b_k)$.

Proof. Choose n and define, by induction on $n-k$, sequences of formulae, χ_l^k, $l = 1,\cdots,m_k$ such that the following conditions hold for any infinite cardinal γ:

(1) $x_1,\cdots,x_k$ are the only free variables of χ_l^k, $l = 1,\cdots,m_k$.

(2) $\models_\gamma \chi_1^k \vee \cdots \vee \chi_{m_k}^k$.

(3) For any $i,j \leqq m_k$, $i \neq j$: $\models_\gamma \neg [\chi_i^k \wedge \chi_j^k]$.

(The notation "$\models_\gamma \chi$" means that for every $\mathfrak{D}$, if $|\mathfrak{D}| \geqq \gamma$ then $\mathfrak{D} \models_\gamma \chi$).

This is done as follows: If $n-k=0$, take all the predicates of L (including the equality-predicate) and perform all the possible substitutions of the variables $x_1,\cdots,x_n$ (part of them or all of them in every possible order). Since L is finite, only a finite number of atomic formulae are obtained in this way, say $P_1,\cdots,P_r$. Put $P_i^0 = P_i$ and $P_i^1 = \neg P_i$ and let ε_i, $1 \leqq i \leqq r$, be 0 or 1. Observe now all the conjunctions of the form: $P_1^{\varepsilon_1} \wedge \cdots \wedge P_r^{\varepsilon_r}$. Arrange them and denote the obtained sequence by $\chi_1^n,\cdots,\chi_{m_n}^n$, $m_n = 2^r$. It is easy to see that this sequence fulfills conditions (1)–(3) above.

Suppose now that the sequence $\chi_1^k,\cdots,\chi_{m_k}^k$ is defined, $1 \leqq k \leqq n$. The sequence $\chi_1^{k-1},\cdots,\chi_{m_{k-1}}^{k-1}$, $m_{k-1} = 3^{m_k}$ will be a sequence of all the possible conjuctions of the form $\psi_1 \wedge \cdots \wedge \psi_{m_k}$ where ψ_j, $1 \leqq j \leqq m_k$, is one of the following: $Qx_k\chi_j^k$, $\neg \exists x_k\chi_j^k$, $\exists x_k\chi_j^k \wedge \neg Qx_k\chi_j^k$. Again it is easy to see that the sequence defined above fulfills conditions (1)–(3).

Now we come to the proof of the lemma by induction on $n-k$ for two fixed models $\mathfrak{A},\mathfrak{B}$ and elements $a_1,\cdots,a_k$ in $\mathfrak{A}$, $b_1,\cdots,b_k$ in $\mathfrak{B}$. Note first that because of (1)–(3) above, there exist unique i, j such that $\mathfrak{A} \models_\alpha \chi_i^k(a_1,\cdots,a_k)$ $\mathfrak{B} \models_\beta \chi_j^k(b_1,\cdots,b_k)$.

Suppose that $n-k=0$ and there exists j such that $\mathfrak{A} \models_\alpha \chi_j^n(a_1,\cdots,a_n)$ and $\mathfrak{B} \models_\beta \chi_j^n(a_1,\cdots,a_n)$. As a result of the definition of χ_l^n, $l = 1,\cdots,m_n$, it follows immediately that the correspondence: $a_i \leftrightarrow b_i$, $i = 1,\cdots,n$, is an isomorphism. So $\mathfrak{A},\mathfrak{B}$ are α,β,n-locally isomorphic over $\langle a_i,b_i\rangle$, $i = 1,\cdots,n$. On the other hand, if $\mathfrak{A},\mathfrak{B}$ are α,β,n-locally isomorphic over $\langle a_i,b_i\rangle$, $i = 1,\cdots,n$, then $\mathfrak{A} \models_\alpha \chi_l^n(a_1,\cdots,a_n)$ iff $\mathfrak{B} \models_\beta \chi_l^n(b_1,\cdots,b_n)$, so by (3) above it follows that there exists j such that $\mathfrak{A} \models_\alpha \chi_j^n(a_1,\cdots,a_n)$ and $\mathfrak{B} \models_\beta \chi_j^n(b_1,\cdots,b_n)$.

Suppose now that the lemma is proved for $n-k$, $1 \leqq k \leqq n$, and prove it for $n-(k-1)$. Suppose first that there exists j, $1 \leqq j \leqq m_{k-1}$, such that

$$\mathfrak{A} \models_\alpha \chi_j^{k-1}(a_1,\cdots,a_{k-1}),\ \mathfrak{B} \models_\beta \chi_j^{k-1}(b_1,\cdots,b_{k-1}). \tag{4}$$

Define sets A_l, B_l, $l = 1,\cdots,m_k$ by the following:

$$A_l = \{a : \mathfrak{A} \models_\alpha \chi_l^k(a_1,\cdots,a_{k-1},a)\}$$

$$B_l = \{b : \mathfrak{B} \models_\beta \chi_l^k(b_1,\cdots,b_{k-1},b)\}\ .$$

By (2), (3) it follows immediately that $\mathfrak{A} = \bigcup_{l=1}^{m_k} A_l$, $\mathfrak{B} = \bigcup_{l=1}^{m_k} B_l$ and $A_i \cap A_l$, $B_i \cap B_l = \varnothing$ for every i,l, $i \neq l$. Suppose that χ_j^{k-1} in (4) is $\psi_1 \wedge \cdots \wedge \psi_{m_k}$, where ψ_i, $1 \leqq i \leqq m_k$, are the formulae of the type mentioned in the definition of the sequence χ_j^{k-1}, $l = 1,\cdots,m_{k-1}$. By (4) we obtain immediately that for every l, $1 \leqq l \leqq m_k$:

$$|A_l| = \alpha \text{ iff } |B_l| = \beta \text{ iff } \psi_l = Qx_k\chi_l^k$$

$$1 \leqq |A_l| < \alpha \text{ iff } 1 \leqq |B_l| < \beta \text{ iff } \psi_l = \exists x_k\chi_l^k \wedge \neg Qx_k\chi_l^k$$

$$A_l = \varnothing \text{ iff } B_l = \varnothing \text{ iff } \psi_l = \neg\exists x_k\chi_l^k.$$

Let F' be the family of all functions from $\mathfrak{A}$ to $\mathfrak{B}$ such that for every $f' \in F'$ and every l, $1 \leqq l \leqq m_k$, we have $f'(A_l) \subseteq B_l$. Let g be any function from $\mathfrak{B}$ to $\mathfrak{A}$ such that $g(B_l) \subseteq A_l$ for every $l \leqq m_k$. Define F by the equality, $F = \{f' \cup g : f' \in F'\}$. Prove now that F fulfills the conditions mentioned in Lemma 1.2 and obtain that $\mathfrak{A},\mathfrak{B}$ are α,β,n-locally isomorphic over $\langle a_i,b_i\rangle$, $i = 1,\cdots,k-1$. It is clear that F fulfills condition (b) (1) in Definition 1.1 (remember that we always suppose that the domain of $\mathfrak{A}$ and $\mathfrak{B}$ are disjoint). Let $A' \subseteq \mathfrak{A}$ be any set such that $|A'| = \alpha$. There exists l such that $|A' \cap A_l| = \alpha$. Denote $A'_l = A' \cap A_l$. Since F' is the family of all functions from $\mathfrak{A}$ to $\mathfrak{B}$ that fulfill certain conditions, it is clear that there exists a function f' in F' such that $|f'(A')| = \beta$. Denote $f = f' \cup g$. So f is in F and $|f(A')| = \beta$. Therefore F fulfills condition (b) (2) in Definition 1.1. Now let $B' \subseteq \mathfrak{B}$ be any set such that $|B'| = \beta$. There exists l such that $|B' \cap B_l| = \beta$. Denote $B'_l = B' \cap B_l$. Again since F' is the family of all functions that fulfills certain conditions there exists $f' \in F'$ such that $f'(A_l) \subseteq B'_l$ and certainly $f'(A_l) \subseteq B'$. But $|B_l| = \beta$ so $|A_l| = \alpha$ and we see immediately that F fulfills also condition (b) (3) in Definition 1.1. Finally, let $a_k \in \mathfrak{A}$, $b_k \in \mathfrak{B}$ be two elements such that $b_k = f(a_k)$ or $a_k = f(b_k)$ for certain f in F. There exists $l \leqq m_k$ such that $a_k \in A_l$. By the definitions of A_l, B_l and F, we obtain $\mathfrak{A} \models_\alpha \chi_l^k(a_1,\cdots,a_{k-1},a_k)$ and $\mathfrak{B} \models_\beta \chi_l^k(b_1,\cdots,b_{k-1},b_k)$. By the induction hypothesis it follows that $\mathfrak{A},\mathfrak{B}$ are α,β,n-locally isomorphic over $\langle a_i, b_i\rangle$, $i = 1,\cdots,k$.

Now suppose that:

(5) $\mathfrak{A} \models_\alpha \chi_j^{k-1}(a_1,\cdots,a_{k-1})$, $\mathfrak{B} \models_\beta \neg\, \chi_j^{k-1}(b_1,\cdots,b_{k-1})$ and prove that $\mathfrak{A},\mathfrak{B}$ are not α,β,n-locally isomorphic over $\langle a_i, b_i\rangle$, $i = 1,\cdots,k-1$. Suppose on the contrary that $\mathfrak{A},\mathfrak{B}$ are α,β,n-locally isomorphic over $\langle a_i, b_i\rangle$, $i = 1,\cdots,k-1$. Then, by Lemma 1.2, there exists a family of functions F that fulfills the conditions of the lemma. Suppose that $\chi_j^{k-1} = \psi_1 \wedge \cdots \wedge \psi_{m_k}$, where ψ_i, $1 \leqq i \leqq m_k$, are the formulae of the type mentioned in the definition of the sequence χ_l^{k-1}, $l = 1,\cdots,m_{k-1}$. By (5) above there exists p such that

(6) $\mathfrak{B} \models_\beta \neg\, \psi_p(b_1,\cdots,b_{k-1})$.

Distinguish between three cases according to the structure of ψ_p:

Case 1. $\psi_p = \neg\, \exists x_k \chi_p^k$. By (6) there exists $b_k \in \mathfrak{B}$ such that $\mathfrak{B} \models_\beta \chi_p^k(b_1,\cdots,b_{k-1},b_k)$. Let f be any function in F. Denote $a_k = f(b_k)$. $\mathfrak{A},\mathfrak{B}$ are α,β,n-locally isomorphic over $\langle ai, bi\rangle$, $i = 1,\cdots,k$, by the assumption on F. Since $\mathfrak{B} \models_\beta \chi_p^k(b_1,\cdots,b_{k-1},b_k)$, it follows by the induction hypothesis that $\mathfrak{A} \models_\alpha \chi_p^k(a_1,\cdots,a_{k-1},a_k)$; hence $\mathfrak{A} \models_\alpha \exists x_k \chi_p^k(a_1,\cdots,a_{k-1},x_k)$ and $\mathfrak{A} \models_\alpha \neg\, \psi_p(a_1,\cdots,a_{k-1})$. Therefore, $\mathfrak{A} \models_\alpha \neg\, \chi_j^{k-1}(a_1,\cdots,a_{k-1})$. This contradicts (5).

Case 2. $\psi_p = \exists x_k \chi_p^k \wedge \neg\, Q x_k \chi_p^k$. By (6) it follows that either $\mathfrak{B} \models_\beta \neg\, \exists x_k \chi_p^k (b_1,\cdots,b_{k-1},x_k)$ or $\mathfrak{B} \models_\beta Q x_k \chi_p^k(b_1,\cdots,b_{k-1},x_k)$. In the first subcase, use arguments similar to those used in Case 1, changing the role of $\mathfrak{A}$ and $\mathfrak{B}$. In the second subcase, denote $B' = \{b\colon\ \mathfrak{B} \models_\beta \chi_p^k(b_1,\cdots,b_{k-1},b)\}$. It is clear that $|B'| = \beta$. Since F fulfills condition (b) (3) in Definition 1.1 there exist $f \in F$ and $A' \subseteq \mathfrak{A}$ such that $|A'| = \alpha$ and $f(A') \subseteq B'$. For every $a_k \in A'$ denote $b_k = f(a_k)$. By Lemma 1.2 it follows that $\mathfrak{A},\mathfrak{B}$ are α,β,n-locally isomorphic over $\langle a_i, b_i\rangle$, $i = 1,\cdots,k$ for every $a_k \in A'$. By the induction hypothesis (since $\mathfrak{B} \models_\beta \chi_p^k(b_1, \cdots,b_{k-1},b)$ for every $b \in B'$) we obtain $\mathfrak{A} \models_\alpha \chi_p^k(a_1,\cdots,a_{k-1},a_k)$ for every $a_k \in A'$. Hence $\mathfrak{A} \models_\alpha Q x_k \chi_p^k(a_1,\cdots,a_{k-1},x_k)$, a contradiction.

Case 3. $\psi_p = Q x_k \chi_p^k$. Define $A' = \{a_k\colon\ \mathfrak{A} \models_\alpha \chi_p^k(a_1,\cdots,a_{k-1},a_k)$. It is clear that $|A'| = \alpha$. By condition (b) (2) in Definition 1.1, there exists $f \in F$ such that $|f(A')| = \beta$. For every $a_k \in A'$ denote $b_k = f(a_k)$. By Lemma 1.2, we obtain again that $\mathfrak{A},\mathfrak{B}$ are α,β,n-locally isomorphic over $\langle a_i, b_i\rangle$, $i = 1,\cdots,k$ for every $a_k \in A$. Therefore, by the induction hypothesis, $\mathfrak{B} \models_\beta \chi_p^k(b_1\ \cdots,b_{k-1},b_k)$, for every $a_k \in A'$, namely, $\mathfrak{B} \models_\beta Q x_k \chi_r^k(b_1,\cdots,b_{k-1},x_k)$, a contradiction.

LEMMA 1.7. *Let L be finite. Then for every n there exists a sequence of sentences* $\chi_1, \cdots, \chi_{m_n}$ *in L(Q) such that for every A, B (models for L) the following holds: A necessary and sufficient condition for A, B to be α, β, n-locally isomorphic is that* $\mathfrak{A} \models_\alpha \chi_i$ *iff* $\mathfrak{B} \models_\beta \chi_i$ *for every* $i \leqq m_n$.

PROOF. Choose n and let $\chi_1, \cdots, \chi_{m_n}$ be the sequence $\chi_1, \cdots, \chi_{m_0}$ defined in the proof of Lemma 1.6. Now use properties (2), (3) of the sequence and the lemma itself.

THEOREM 1.8. *Let L be finite. Then* $\mathfrak{A}^\alpha \equiv^\beta \mathfrak{B}$ *iff* $\mathfrak{A}, \mathfrak{B}$ *are α, β, n-locally isomorphic for every n.*

PROOF. The result is obtained by using Theorem 1.5 and Lemma 1.7.

REMARK 1.1. If $\alpha = \beta$ in the discussion above, then it is sufficient to demand that every F in Definition 1.1 be a singleton and replace conditions (b) (2), (3) by the following condition: For every $X \subseteq \mathfrak{A} \cup \mathfrak{B}$ such that $|X| = \alpha$, also $|f(X)| = \alpha$.

Call the concepts defined by these modifications by: α, n-extension of $\langle a_i, b_i \rangle$, $i = 1, \cdots, k$, α, n-local isomorphism over $\langle a_i, b_i \rangle$, $i = 1, \cdots, k$ and α, n-local isomorphism. All the lemmas and the theorems remain true after replacing the old concepts by the new ones and the proofs become simpler.

The new concepts above are equivalent to the game theoretic concepts of Lipner [5].

REMARK 1.2. If $\mathfrak{A}, \mathfrak{B}$ are α, β, n-locally isomorphic and $n > 0$ then $|\mathfrak{A}| = \alpha$ iff $|\mathfrak{B}| = B$. If $|\mathfrak{A}| < \alpha$, $|\mathfrak{B}| < \beta$ then it is sufficient to demand that every F in Definition 1.1 be a singleton that fulfills only (b) (1) and (b) (4) of the definition. In this case $\mathfrak{A}^\alpha \equiv^\beta \mathfrak{B}$ iff $\mathfrak{A} \equiv \mathfrak{B}$ and $\mathfrak{A}^\alpha \equiv {}^\beta_n \mathfrak{B}$ iff $\mathfrak{A} \equiv {}_n \mathfrak{B}$ where "$\mathfrak{A} \equiv {}_n \mathfrak{B}$" means: For every sentence ϕ in $L \cap P_{n,0}(Q)$ the following holds: $\mathfrak{A} \models \phi$ iff $\mathfrak{B} \models \phi$.

The concepts obtained in this case are equivalent to the game theoretic concepts of Ehrenfeucht [1].

2. The theory of one equivalence relation

Let α be any infinite cardinal and let S be a set of sentences in $L(Q)$. We say that $\mathfrak{A}$ is an α-model for S if $\mathfrak{A} \models_\alpha \phi$ for every $\phi \in S$. This will be denoted by $\mathfrak{A} \models_\alpha S$.

Now let T be any first order theory (namely, a theory formulated in L). Denote $T(Q) = T \cup \{Qx[x = x]\}$ and define $T(\alpha) = \{\phi : \phi$ is a sentence in $L(Q)$ such that for every $\mathfrak{A}$, if $\mathfrak{A} \models_\alpha T(Q)$ then $\mathfrak{A} \models_\alpha \phi\}$.

In this section T will be the first order theory of one equivalence relation (the language for T contains therefore only one binary predicate $\sim$ in addition to the equality predicate). The main result of this section is that $T(\alpha) = T(\aleph_0)$ for every α and $T(\aleph_0)$ is decidable.

LEMMA 2.1. *Let α,β be two regular cardinals, α, $\beta \geqq \aleph_1$. Then for every α-model $\mathfrak{A}$ of $T(Q)$ there exists a β-model $\mathfrak{B}$ of $T(Q)$ such that $\mathfrak{A}^\alpha \equiv {}^\beta\mathfrak{B}$.*

PROOF. Suppose that $\alpha > \beta$ and $\mathfrak{A} \vDash_\alpha T(Q)$. By Fuhrken [2], it is possible to suppose w.l.g. that $|\mathfrak{A}| = \alpha$. The equivalence relation on $\mathfrak{A}$ divides the domain of $\mathfrak{A}$ into disjoint equivalence classes. There exists an ordinal μ equal or smaller than the ordinal-product $\alpha \cdot 3$ such that we may suppose:

(1) For every $\gamma < \mu$, A_γ is an equivalence class in $\mathfrak{A}$ and $\mathfrak{A} = \cup_{\gamma<\mu} A_\gamma$.

(2) There exist two ordinals μ_1, μ_2, $0 \leqq \mu_1 \leqq \mu_2 \leqq \mu$ such that:

(a) $|A_\gamma| < \aleph_0$ for every $\gamma < \mu_1$.

(b) $\aleph_0 \leqq |A_\gamma| < \alpha$ for every γ, $\mu_1 \leqq \gamma < \mu_2$.

(c) $|A_\gamma| = \alpha$ for every γ, $\mu_2 \leqq \gamma < \mu$. Define now sets B_γ, $\gamma < \nu$, where ν is an ordinal number equal or smaller than the ordinal product $\beta \cdot 3$ and will be fixed later. All the sets should be disjoint to the domain of $\mathfrak{A}$.

Step 1. For every $n < \aleph_0$, denote $\kappa_n = |\{\gamma\colon |A_\gamma| = n\}|$. If $\kappa_n < \aleph_0$, define κ_n disjoint sets of power n. If $\aleph_0 \leqq \kappa_n < \alpha$, define $\aleph_0$ disjoint sets of power n. If $\kappa_n = \alpha$, define β disjoint sets of power n.

Step 2. Denote $\kappa' = |\{\gamma : \aleph_0 \leqq |A_\gamma| < \alpha\}|$. If $\kappa' < \aleph_0$, define κ' disjoint sets of power $\aleph_0$. If $\aleph_0 \leqq \kappa' < \alpha$, define $\aleph_0$ disjoint sets of power $\aleph_0$. If $\kappa' = \alpha$, define β disjoint sets of power $\aleph_0$.

Step 3. Denote $\kappa = |\{\gamma : |A_\gamma| = \alpha\}|$. If $\kappa < \aleph_0$, define κ disjoint sets of power β. If $\aleph_0 \leqq \kappa < \alpha$, define $\aleph_0$ disjoint sets of power β and if $\kappa = \alpha$, define β disjoint sets of power β.

The sets should be defined in such a way that the sets which are defined in a later step are also disjoint to all sets defined previously. We may suppose that there exist three ordinals ν_1, ν_2, ν, $0 \leqq \nu_1 \leqq \nu_2 \leqq \nu \leqq \beta \cdot 3$ such that:

(3) B_γ is one of the sets defined above for every $\gamma < \nu$.

(4) (a) $|B_\gamma| < \aleph_0$ for every $\gamma < \nu_1$.

(b) $|B_\gamma| = \aleph_0$ for every γ, $\nu_1 \leqq \gamma < \nu_2$.

(c) $|B_\gamma| = \beta$ for every γ, $\gamma_2 \leqq \gamma < \nu$.

$\mathfrak{B}$ will be a structure, the domain of which is $\cup_{\gamma<\nu} B_\gamma$, and for every $b_1, b_2 \in \mathfrak{B}$, $\mathfrak{B} \models b_1 \sim b_2$ iff there exists $\gamma < \nu$ such that $b_1, b_2 \in B_\gamma$. By the regularity of α, we obtain $|\mathfrak{B}| = \beta$.

Now we shall show by induction on n that $\mathfrak{A}, \mathfrak{B}$ are α, β, n-locally isomorphic for every n. We use Lemma 1.4. Note first that $\mathfrak{A}, \mathfrak{B}$ are $\alpha, \beta, 0$-locally isomorphic by definition. Suppose that there exists an α, β, n-extension Γ of the empty sequence such that for every sequence $\langle a_i, b_i \rangle$, $i = 1, \cdots, n$ in Γ, we have:

(5) (a) For every $m < \aleph_0$, a_i belongs to an equivalence-class of power m iff b_i belongs to an equivalence class of power m.

(b) a_i belongs to an equivalence-class of infinite power smaller than α iff b belongs to an equivalence-class of power $\aleph_0$.

(c) a_i belongs to an equivalence-class of power α iff b_i belongs to an equivalence-class of power β.

For every $\langle a_i, b_i \rangle$, $i = 1, \cdots, n$ in Γ, we should define now a family of functions F that fulfills conditions (b) (1)–(4) in Definition 1.1 and conditions (5) (a)–(c) above. Although one can easily convince himself that there exists such an F, by observation of the construction of $\mathfrak{B}$, we shall write the definition in full detail for the sake of formality. (We are aware of the fact that these details are a bit tedious.)

Let h_1 be a function from μ to ν. For every $\gamma < \mu$ denote $\gamma' = h_1(\gamma)$. Observe such an h_1 that fulfills (6)–(9) below:

(6) If $a_i \in A_\gamma$ for certain $i \leqq n$ then $b_i \in B_{\gamma'}$. Denote $M_1 = \{h_1(\gamma): \gamma < \mu$ and there exists $i \leqq n$ such that $a_i \in A_\gamma\}$.

In (7)–(9) below suppose that $a_i \notin A_\gamma$ for every $i \leqq n$.

(7) If $\gamma < \mu$, then $\gamma' < \nu_1$, $|A_\gamma| = |B_{\gamma'}|$ and $\gamma' \notin M_1$.

(8) If $\mu_1 \leqq \gamma < \mu_2$ then $\nu_1 \leqq \gamma' < \nu_2$ and $\gamma' \notin M_1$.

(9) If $\mu_2 \leqq \mu_1 < \mu$ then $\nu_2 \leqq \gamma' < \nu$ and $\gamma' \notin M_1$.

By the definition of $\mathfrak{B}$ and by (5) (a)–(c), it is easy to see that there exist functions that satisfy (6)–(9) above. Let h_1 be such a function. Let $f_{1,\gamma}^{h_1}$, $\gamma < \mu$, be a transfinite sequence of functions such that:

(10) $f_{1,\gamma}^{h_1}$ is a function from A_γ on $B_{h_1(\gamma)}$ and for every $\gamma < \mu$ and $i \leqq n$ the following holds: If $a_i \in A_\gamma$ then $f_{1,\gamma}^{h_1}(a_i) = b_i$ and if $a \neq a_i$ then $f_{1,\gamma}^{h_1}(a) \neq b_i$.

Define:

(11) $f^{h_1} = \bigcup_{\gamma<u} f_{1,\gamma}^{h_1}$.

Let F^{h_1} be the set of all functions obtained by (11) letting $f_{1,\gamma}^{h_1}$, $\gamma < \mu$, run over all possible sequences that satisfy (10) for a fixed h_1.

Define: $F_1 = \cup\{F^{h_1}: h_1$ runs over all possible functions that fulfill (6)–(9)$\}$.

Now let h_2 be a function from ν to μ. For every $\gamma < \nu$ denote $\gamma' = h_2(\gamma)$. Observe such an h_2 that fulfills conditions (12)–(15) below:

(12) If $b_i \in B_\gamma$ for certain $i \leqq n$ then $a_i \in A_{\gamma'}$. Denote $M_2 = \{h_2(\gamma): \gamma < \nu$ and there exists $i \leqq n$ such that $b_i \in B_\gamma\}$. In (13)-(15) below, suppose that $b_i \notin B_\gamma$ for every $i \leqq n$.

(13) If $\gamma < \nu_1$ then $\gamma' < \mu_1$, $|B_\gamma| = |A_{\gamma'}|$ and $\gamma' \notin M_2$.

(14) If $\nu_1 \leqq \gamma < \nu_2$ then $\mu_1 \leqq \gamma' < \mu_2$ and $\gamma' \notin M_2$.

(15) If $\nu_2 \leqq \gamma < \nu$ then $\mu_2 \leqq \gamma' < \mu_2$ and $\gamma' \notin M_2$. Again, by the definition of $\mathfrak{B}$ and by (5) (a)–(c), there exists such an h_2.

Let $f_{2,\gamma}$, $\gamma < \nu$, be a sequence of functions such that $f_{2,\gamma}$ is a function from B_γ into $A_{h_2(\gamma)}$ and for every $\gamma < \nu$ and $i \leqq n$, if $b_i \in B_\gamma$ then $a_i = f_{2,\gamma}(b_i)$ and if $b \neq b_i$ then $f_{2,\gamma}(b) \neq a_i$. Denote $g = \bigcup_{\gamma<\nu} f_{2,\gamma}$. The desired family of functions will be defined by: $F = \{f_1 \cup g: f_1 \in F_1\}$.

It is clear that F fulfills condition (b) (1) of Definition 1.1.

Let $A' \subseteq \mathfrak{A}$ be any set such that $|A'| = \alpha$. If there exists $\gamma < \mu$ such that $|A' \cap A_\gamma| = \alpha$, then $|A_\gamma| = \alpha$ and for every h_1 that fulfills (6)–(9) above, we have $|B_{h_1(\gamma)}| = \beta$. By the definition of F^{h_1}, there exists f^{h_1} in F^{h_1} such that $|f^{h_1}(A' \cap A_\gamma)| = \beta$. Therefore there also exists f in F such that $|f(A' \cap A_\gamma)| = \beta$ and since $\beta = |f(A' \cap A_\gamma)| \leqq |f(A')| \leqq \beta$, we obtain $f(A') = \beta$. If, on the other hand, $|A' \cap A_\gamma| < \alpha$ for every $\gamma < \mu$, denote $M' = \{\gamma: \gamma < \mu, A_\gamma \cap A' \neq \varnothing\}$. $|M'| = \alpha$ because of the regularity of α. There exists h_1 that satisfies (6)–(9) above such that $|h_1(M')| = \beta$ and, for this h_1, every f^{h_1} in F^{h_1} satisfies $|f^{h_1}(A')| = \beta$. So it is clear that there exists f in F such that $|f(A')| = \beta$ and we have proved that F fulfills also condition (b) (2) in Definition 1.1.

Let $B' \subseteq \mathfrak{B}$ be any set such that $|B'| = \beta$. If there exists $\gamma' < \nu$ such that $|B' \cap B_{\gamma'}| = \beta$ then $|B_{\gamma'}| = \beta$. Therefore there exists γ, $\mu_2 < \gamma < \mu$, and there exists h_1 such that $h_1(\gamma) = \gamma'$. Observe F^{h_1}. Certainly it contains a function f^{h_1} such that $f^{h_1}(A_\gamma) \subseteq B' \cap B_{\gamma'}$. So there also exists f in F such that $f(A_\gamma) \subseteq B' \cap B_{\gamma'}$. Denote $A' = A_\gamma$ and obtain that, in this case, F fulfills condition (b) (3) of Definition 1.1. If, on the other hand, $|B' \cap B_\gamma| < \beta$ for every $\gamma < \nu$ denote: $M'' = \{\gamma: \gamma < \nu, B_\gamma \cap B' \neq \varnothing\}$. Since β is regular, then $|M''| = \beta$. There exists a set $M' \subseteq \mu$, $|M'| = \alpha$, and there exists h_1 that fulfills (6)–(9) above such that $h_1(M') \subseteq M''$. Likewise, there exist functions $f_{1,\gamma}^{h_1}$, $\gamma \in M'$, that fulfill (10) above and there exist sets $A'_\gamma \subseteq A_\gamma$, $\gamma \in M'$, that fulfill $A_\gamma \neq \emptyset$ and $f_{1,\gamma}^{h_1}(A'_\gamma) \subseteq B_{h_1(\gamma)} \cap B'$ for every $\gamma \in M'$. Therefore there exists f in F such that $f(\bigcup_{\gamma \in M'} A'_\gamma) \subseteq \bigcup_{\gamma \in M'} B_{h_1(\gamma)} \cap B' \subseteq B'$. Denote $A' = \bigcup_{\gamma \in M'} A'_\gamma$ and obtain that, also in this case, F fulfills condition (b) (3) of Definition 1.1.

Finally, if $b_{n+1} = f(a_{n+1})$ or $a_{n+1} = f(b_{n+1})$ for certain f in F, we obtain immediately by the definition of F that the correspondence: $a_i \leftrightarrow b_i$, $i = 1, \cdots, n+1$, is an isomorphism and the sequence $\langle a_i, b_i \rangle$, $i = 1, \cdots, n+1$, fulfills also (5) (a)–(c) above. So we have finished the induction step and obtained that $\mathfrak{A}, \mathfrak{B}$ are α, β, n-locally isomorphic for every n. Hence, by Theorem 1.5 we have $\mathfrak{A}^\alpha \equiv^\beta \mathfrak{B}$.

By the above considerations it is clear how to construct a model $\mathfrak{B}$ of power β such that $\mathfrak{A}^\alpha \equiv^\beta \mathfrak{B}$ in case that $\beta > \alpha$.

LEMMA 2.2. *Let α be a regular uncountable cardinal. Then for every α-model $\mathfrak{A}$ of $T(Q)$ and for every positive integer m there exists an $\aleph_0$-model $\mathfrak{B}$ of $T(Q)$ such that $\mathfrak{A}^\alpha \equiv_m^{\aleph_0} \mathfrak{B}$.*

PROOF. The proof of this lemma differs from the proof of Lemma 2.1 only in details. So for a given m and a given $\mathfrak{A}$, we shall explain how to construct a suitable $\mathfrak{B}$ but we shall leave it to the reader to show that $\mathfrak{A}^\alpha \equiv_m^{\aleph_0} \mathfrak{B}$.

Suppose $|A| = \alpha$ and, as in the proof of Lemma 2.1, suppose also that there exist three ordinals μ_1, μ_2, μ, $0 \leqq \mu_1 \leqq \mu_2 \leqq \mu \leqq \alpha \cdot 3$, and there exist mutually disjoint sets A_γ, $\gamma < \mu$, such that (1) and (2) in the proof of Lemma 2.1 hold. Define sets B_γ, $\gamma < \nu$, where $\nu \leqq \omega \cdot 3$ and will be fixed later.

Step 1. Let κ_n be exactly as in Step 1 of Proof 2.1. For every n such that $n \leqq m$, if $\kappa_n \leqq m$, define κ_n disjoint sets of power n; if $m < \kappa_n < \alpha$, define $m+1$ disjoint sets of power n and if $\kappa_n = \alpha$, define $\aleph_0$ disjoint sets of power n. On the other hand, for every n such that $n > m$, if $\kappa_n = \alpha$, define $\aleph_0$ disjoint sets of power $m+1$. After this, observe $\sum(\kappa_n : n < m, \kappa_n \neq \alpha)$. If it is smaller than $m+1$, define for every $n > m$ (such that $\kappa_n \neq \alpha$) κ_n disjoint sets of power $m+1$. But if $\sum(\kappa_n : n > m, \kappa_n \neq \alpha) > m$, define $m+1$ disjoint sets of power $m+1$ (note that $\sum(\kappa_n : n > m, \kappa_n \neq \alpha) < \alpha$ since α is regular and uncountable).

Step 2. Let κ' be exactly as in Step 2 of Proof 2.1. If $\kappa' \leqq m$, define κ' disjoint sets of power $m+1$; if $m < \kappa' < \alpha$, define $m+1$ disjoint sets of power $m+1$; and if $\kappa' = \alpha$, define $\aleph_0$ disjoint sets of power $m+1$.

Step 3. Let κ be exactly as in Step 3 of Proof 2.1. If $\kappa \leqq m$, define κ disjoint sets of power $\aleph_0$; if $m < \kappa < \alpha$, define $m+1$ disjoint sets of power $\aleph_0$; and if $\kappa = \alpha$, define $\aleph_0$ disjoint set of power $\aleph_0$.

The sets should be defined in such a way that the sets which are defined in a later step are also disjoint to all sets defined in previous steps and to the domain of $\mathfrak{A}$.

We may suppose that there exist three ordinals, ν_1, ν_2, ν, $0 \leqq \nu_1 \leqq \nu_2 \leqq \nu \leqq \omega \cdot 3$, such that:

(a) B_γ is one of the sets defined above for every $\gamma < \nu$.

(b) For every $\gamma < \nu_1$, B_γ is finite and was defined in Step 1 above.

(c) For every γ, $\nu_1 \leqq \gamma < \nu_2$, B_γ is of power $m+1$ and was defined in Step 2 above.

(d) For every γ, $\nu_2 \leqq \gamma < \nu$, B_γ is of power $\aleph_0$. $\mathfrak{B}$ will be a model the domain of which is $\bigcup_{\gamma<\nu} B_\gamma$ and for every $b_1, b_2 \in \mathfrak{B}$: $\mathfrak{B} \models b_1 \sim b_2$ iff there exists $\gamma < \nu$ such that $b_1, b_2 \in B_\gamma$.

As mentioned before, we leave it to the reader to show that $\mathfrak{A}, \mathfrak{B}$ are $\alpha, \aleph_0, k$-locally isomorphic for every $k \leqq m$. This is done by induction on k.

REMARK 2.1. Lemma 2.2 cannot be improved because for any $\alpha > \aleph_0$, it is not true that if $\mathfrak{A}$ is an α-model of $T(Q)$ then there exists an $\aleph_0$-model of $T(Q)$ such that $\mathfrak{A}^\alpha \equiv^{\aleph_0} \mathfrak{B}$. To show this, define a set of sentences S by the following:

$$S = \{\forall x \neg Qy[x \sim y]\} \cup \{\forall x \exists^{\geqq n} z[z \sim x] : n < \aleph_0\},$$

where $\exists^{\geqq n} z[z \sim x]$ is a formula in the language of T which "says" that there exist at least n different elements equivalent to x.

It is easy to see that for every $\alpha > \aleph_0$, there exists $\mathfrak{A}$ such that $\mathfrak{A} \models_\alpha S \cup T(Q)$ but there is no $\mathfrak{B}$ such that $\mathfrak{B} \models_{\aleph_0} S \cup T(Q)$.

On the other hand, for every $\aleph_0$-model $\mathfrak{B}$ of $T(Q)$ and for every α, there exists an α-model $\mathfrak{A}$ of $T(Q)$ such that $\mathfrak{A}^\alpha \equiv^{\aleph_0} \mathfrak{B}$. This is an immediate result of a general theorem by Fuhrken [3]. So there is no point of mentioning it in the lemma.

LEMMA 2.3. *Let α be a singular cardinal with cofinality greater than $\aleph_0$. Denote $\beta = \mathrm{cf}(\alpha)$. Then for every α-model $\mathfrak{A}$ of $T(Q)$ there exists $\mathfrak{B}$ such that $\mathfrak{A}^\alpha \equiv^\beta \mathfrak{B}$ and for every β-model $\mathfrak{B}$ of $T(Q)$ there exists $\mathfrak{A}$ such that $\mathfrak{A}^\alpha \equiv^\beta \mathfrak{B}$.*

PROOF. Again, the proof of this lemma differs from the proof of Lemma 2.1 only in details. So we shall only explain how, for a given $\mathfrak{A}$, one should construct a suitable $\mathfrak{B}$ but we shall leave it to the reader to show that $\mathfrak{A}^{\alpha} \equiv {}^{\beta}\mathfrak{B}$.

Let μ, $A_\gamma(\gamma < \mu)$, $\mu_1, \mu_2, \kappa_n(n < \aleph_0)$, κ', κ be as in the proof of Lemma 2.1.

Define sets B_γ, $\gamma < \nu$, where $\nu \leqq \beta \cdot 3$ and will be defined later.

Step 1. For every $n < \aleph_0$, if $\kappa_n < \aleph_0$, define κ_n disjoint sets of power n; if $\aleph_0 \leqq \kappa_n < \alpha$, define $\aleph_0$ disjoint sets of power n; and if $\kappa_n = \alpha$, define β disjoint sets of power n.

Step 2. If $\kappa' < \mathrm{cf}(\alpha)$, define $\min(\kappa', \aleph_0)$ disjoint sets of power $\aleph_0$. If $\kappa' \geqq \mathrm{cf}(\alpha)$ and also $\Sigma(|A_\gamma| : \aleph_0 \leqq |A_\gamma| < \alpha) = \alpha$, define β disjoint sets of power $\aleph_0$. If $\kappa' \geqq \mathrm{cf}(\alpha)$ but $\Sigma(|A_\gamma| : \aleph_0 \leqq |A_\gamma| < \alpha) < \alpha$, define $\aleph_0$ disjoint sets of power $\aleph_0$.

Step 3. If $\kappa < \mathrm{cf}(\alpha)$, define $\min(\kappa, \aleph_0)$ disjoint sets of power β. If $\kappa \geqq \mathrm{cf}(\alpha)$, define β disjoint sets of power β.

The definitions of $\nu, B_\gamma(\gamma < \nu)$ and $\mathfrak{B}$ are the same as in the proof of Lemma 2.1.

LEMMA 2.4. *Let α be a singular cardinal, $\mathrm{cf}(\alpha) = \aleph_0$. Then for every α-model $\mathfrak{A}$ of $T(Q)$ and for every positive integer m there exists $\mathfrak{B}$ such that* $\mathfrak{A}^{\alpha} \equiv {}^{\aleph_0}_{m}\mathfrak{B}$.

PROOF. Let $\mathfrak{A}$ be given ($|\mathfrak{A}| = \alpha$) and let μ, μ_1, μ_2, $A_\gamma(\gamma < \mu)$, $\kappa_n(n < \aleph_0)$, κ' and κ be as in the proof of Lemma 2.1. Construct $\mathfrak{B}$ as follows:

Step 1. For every n such that $n \leqq m$, if $\kappa_n \leqq m$, define κ_n disjoint sets of power n. If $m < \kappa_n < \alpha$, define $m + 1$ disjoint sets of power n. If $\kappa_n = \alpha$, define $\aleph_0$ disjoint sets of power n. Now observe $\Sigma_{n>m}\kappa_n$. If $\Sigma_{n>m}\kappa_n = \alpha$, define $\aleph_0$ disjoint sets of power $m + 1$. If $\Sigma_{n>m}\kappa_n < \alpha < \alpha$, define $m + 1$ disjoint sets of power $m + 1$.

Step 2. If $\kappa' < \aleph_0$, define κ' disjoint sets of power $m + 1$. If $\kappa' \geqq \aleph_0$ and also $\Sigma(|A_\gamma| : \aleph_0 \leqq |A_\gamma| < \alpha) = \alpha$, define $\aleph_0$ disjoint sets of power $m + 1$. If $\kappa' \geqq \aleph_0$ but $\Sigma(|A_\gamma| : \aleph_0 \leqq |A_\gamma| < \alpha) < \alpha$, define $m + 1$ disjoint sets of power $m + 1$.

Step 3. Define $\min(\kappa, \aleph_0)$ disjoint sets of power $\aleph_0$.

The definitions of ν, $B_\gamma(\gamma < \nu)$ and $\mathfrak{B}$ are as in the proof of Lemma 2.1.

REMARK 2.2. Like Lemma 2.2, also Lemma 2.4 cannot be improved. To show this, use the same example as in Remark 2.1.

THEOREM 2.1. *For every α, for every positive integer m and for any α-model $\mathfrak{A}$ of $T(Q)$, there exists $\mathfrak{B}$ such that $\mathfrak{A}^{\alpha} \equiv {}^{\aleph_0}_{m}\mathfrak{B}$.*

PROOF. If α is regular or has the cofinality $\aleph_0$, the theorem is an immediate result of Lemma 2.2 or Lemma 2.4. If α is singular and $\mathrm{cf}(\alpha) > \aleph_0$, use Lemmas 2.3, 2.2 and the following general observation:

For every α,β,γ and every $\mathfrak{A},\mathfrak{B},\mathfrak{C}$, if $\mathfrak{A}^{\alpha} \equiv {}^{\beta}\mathfrak{B}$ and $\mathfrak{B}^{\beta} \equiv {}^{\gamma}\mathfrak{C}$ then $\mathfrak{A}^{\alpha} \equiv {}^{\gamma}\mathfrak{C}$.

The following theorem is a result of a remark by S. Shelah.

THEOREM 2.2. *For every $\alpha,\beta > \aleph_0$, if $\mathfrak{A}$ is an α-model of $T(Q)$ then there exists $\mathfrak{B}$ such that $\mathfrak{A}^{\alpha} \equiv {}^{\beta}\mathfrak{B}$.*

PROOF. If α,β are regular, use Lemma 2.1. Suppose β is singular. Let ϕ be any sentence in $L(Q)$ such that $\mathfrak{A} \models_{\alpha} \phi$. By Theorem 2.1, there exists an $\aleph_0$-model for ϕ. Hence, by a theorem of Fuhrken [3], there exists a $\beth_{\omega}$-model for ϕ. Since $\beth_{\omega}$ is a singular strong limit cardinal, we obtain by a compactness theorem of Keisler [4], a $\beth_{\omega}$-model $\mathfrak{B}_1$ such that $\mathfrak{A}^{\alpha} \equiv {}^{\beth_{\omega}}\mathfrak{B}_1$. Again, by Keisler [4], we obtain $\mathfrak{B}$ such that $\mathfrak{B}_1{}^{\beth_{\omega}} \equiv {}^{\beta}\mathfrak{B}$. Therefore, $\mathfrak{A}^{\alpha} \equiv {}^{\beta}\mathfrak{B}$.

Recall now the notation $T(\alpha)$ in the beginning of this section.

THEOREM 2.3. *$T(\alpha) = T(\aleph_0)$ for every α, and $T(\aleph_0)$ is decidable.*

PROOF. By a general theorem of Fuhrken [3], $T(\alpha) \subseteq T(\aleph_0)$ for every α. Suppose $T(\aleph_0) \neq T(\alpha)$ for certain α. Then there exists a sentence ϕ in $L(Q)$ such that $\phi \in T(\aleph_0)$ but $\phi \notin T(\alpha)$. So there exists an α-model $\mathfrak{A}$ of $T(Q)$ such that $\mathfrak{A} \models_{\alpha} \neg \phi$. Without loss of generality, suppose ϕ is in a prenex normal form in $L(Q)$ and its prefix has m quantifiers. By Theorem 2.1., there exists $\mathfrak{B}$ such that $\mathfrak{B} \models_{\aleph_0} \neg \phi$, a contradiction.

The decidability of $T(\aleph_0)$ is obtained by using Rabin's [6] method. For the sake of completeness, we shall describe it briefly. $S2S$ is the second order theory of the structure $\langle T_2, r_0, r_1, \leqq, \preccurlyeq \rangle$ where T_2 is the set of all finite sequences the members of which belong to $\{0,1\}$; r_i, $i = 0,1$, are two functions such that $r_i(x) = xi$ for everty x in T_2; $\leqq$ is a partial ordering defined by: $x \leqq y$ iff the sequence x is a segment of the sequence y; $\preccurlyeq$ is the lexicographic ordering defined in the usual way by the stipulation: $0 \preccurlyeq 1$. In Rabin [6], it is proved that $S2S$ is decidable.

We shall show how to embede a given countable model of T in $\langle T_2, r_0, r_1, \leqq, \preccurlyeq \rangle$. First define in $S2S$, three subsets of T_2 denoted by M_1, M_2 and M. This is done by:

$$x \in M_1 \;\leftrightarrow\; x \preccurlyeq 1 \wedge \exists y[x = r_1(y)]$$

$$x \in M_2 \;\leftrightarrow\; \exists z \exists y \exists u[y = r_1(z) \wedge u = r_0(y) \wedge u \leqq x]$$

$$x \in M \;\leftrightarrow\; x \in M_1 \wedge x \notin M_2.$$

A brief observation of the complete binary tree will make the structure of M clear.

Let $\mathfrak{A}$ be any countable model of T. We can write: $\mathfrak{A} = \bigcup_{0<n<\aleph_0} A_n$, where A_n, $0 < n < \aleph_0$, are the equivalence-classes of $\mathfrak{A}$ (some of them may be empty). There exists a one-to-one mapping g from $\mathfrak{A}$ into M such that A_n (if not empty) is mapped by g into the set $\{\underbrace{0 \cdots 0}_{n}\ \underbrace{1 \cdots 1}_{k} : k = 1,2,\cdots\}$. It is easy to see that for every $a, b \in \mathfrak{A}$, $\mathfrak{A} \models a \sim b$ iff $g(a) \leqq g(b)$ or $g(b) \leqq g(a)$.

On the other hand, every infinite subset $A \subset M$ can be turned into an $\aleph_0$-model $\mathfrak{A}$ of $T(Q)$ by defining $\mathfrak{A} \models a \sim b$ iff $a \leqq b$ or $b \leqq a$.

In Rabin [6], it is proved that the second order unary relation "Y is a finite set" can be defined in $S2S$. Denote by F a formula that defines this relation. Let X, Y be second order variables. To every formula ψ in $L(Q)$ (where L is the first order language for the theory of one equivalence-relation) correspond now a formula $\bar{\psi}(X)$ in the second order language for $\langle T_2, r_0, r_1, \leqq, \preccurlyeq \rangle$. $\bar{\psi}(X)$ will contain X as the only second order free variable iff ψ contains bound variables. The correspondence is defined by induction on ψ.

(1) If ψ is $x \sim y$, then $\bar{\psi}$ is $x \leqq y \vee y \leqq x$. Suppose $\bar{\psi}_1, \bar{\psi}_2$ are defined for given ψ_1, ψ_2.

(2) If ψ is $\psi_1 \vee_2$, then $\bar{\psi}$ is $\bar{\psi}_1 \vee \bar{\psi}_2$.

(3) If ψ is $\neg \psi_1$, then $\bar{\psi}$ is $\neg \bar{\psi}_1$.

(4) If ψ is $\exists v \psi_1$, then $\bar{\psi}$ is $\exists v[v \in X \wedge \bar{\psi}_1(X)]$

(5) If ψ is $Qv\psi_1$, then $\bar{\psi}$ is $\exists Y[\neg F(Y) \wedge Y \subseteq X \wedge \forall v[v \in Y \to \bar{\psi}_1(X)]$.

Finally let ϕ be any sentence in $L(Q)$. It is easy to see that $\phi \in T(\aleph_0)$ iff $\forall X[\neg F(X) \wedge X \subseteq M \to \bar{\phi}(X)]$ is in $S2S$.

We want to conclude this section with a remark on the "expressive power" of $L(Q)$ (where L is as above).

REMARK 2.3. There does not exist a sentence ϕ in $L(Q)$ such that for every α-model $\mathfrak{A}$ of $T(Q)$, $\mathfrak{A} \models_\alpha \phi$ iff there exist infinitely many elements in $\mathfrak{A}$ such that no two of them are equivalent.

To show this, suppose that there exists such a ϕ. Without loss of generality, suppose that ϕ is in a prenex normal form in $L(Q)$ and its prefix contains m quantifiers. Let $\mathfrak{A}$ be an α-model for $T(Q)$ that contains $\aleph_0$ equivalence-classes, each of them containing α elements. Let $\mathfrak{B}$ be an α-model for $T(Q)$ that contains m equivalence-classes, each of them containing α elements. By the methods of this section, it is easy to show that $\mathfrak{A}^{\alpha} \equiv {}^{\alpha}_{m}\mathfrak{B}$, so $\mathfrak{B} \models_{\alpha} \phi$, a contradiction.

3. Theory of the k unary relations

Let k be finite and let L be the first order theory language that contains only k unary predicates $P_1, \cdots, P_k$ (in addition to the logical) constants). Denote $P_i^0 = P_i$ and $P_i^1 = \neg P_i$, $i = 1, \cdots, k$ and observe all the conjunctions of the type $P_1^{\varepsilon_1}(x) \wedge \cdots \wedge P_k^{\varepsilon_k}(x)$, $\varepsilon_i \in \{0,1\}$, $i = 1, \cdots, k$. Denote them by $\psi_j(x)$, $j = 1, \cdots, 2^k$. Let $\phi(x)$ be any formula in L such that x is its only free variable. Let $\mathfrak{A}$ be any model for L. Denote $\phi(\mathfrak{A}) = \{a \colon \mathfrak{A} \models \phi(a)\}$. It is easy to see that $\psi_j(\mathfrak{A}) \cap \psi_i(\mathfrak{A}) = \varnothing$ when $i \neq j$ and $\bigcup_{1 \leqq j \leqq 2^k} \psi_j(\mathfrak{A}) = \mathfrak{A}$. Let $\mathfrak{B}$ be another model for L. Suppose $a_i \in \mathfrak{A}$, $b_i \in \mathfrak{B}$, $i = 1, \cdots, n$. It is easy to see that the correspondence: $a_i \leftrightarrow b_i$, $i = 1, \cdots, n$, is an isomorphism regarding the k unary relations in $\mathfrak{A}, \mathfrak{B}$ iff it is an isomorphism regarding the relations $\psi_j(\mathfrak{A}), \psi_j(\mathfrak{B})$, $j = 1, \cdots, 2^k$.

In this section, T will denote the first order theory of k unary relations.

THEOREM 3.1. *For every $\alpha, \beta > \aleph_0$ and every α-model $\mathfrak{A}$ of $T(Q)$, there exists $\mathfrak{B}$ such that $\mathfrak{A}^{\alpha} \equiv {}^{\beta}\mathfrak{B}$.*

PROOF. Suppose $\mathfrak{A}$ is given and $|\mathfrak{A}| = \alpha$. Let A be the domain of $\mathfrak{A}$ and observe the structure $\langle A, \psi_1(A), \cdots, \psi_{2^k}(\mathfrak{A})\rangle$. Define 2^k mutually disjoint sets B_i, $i = 1, \cdots, 2^k$, by the following:

If $|\psi_i(\mathfrak{A})| < \aleph_0$, then B_i contains $|\psi_i(\mathfrak{A})|$ elements.

If $\aleph_0 \leqq |\psi_i(\mathfrak{A})| < \alpha$, then B_i contains $\aleph_0$ elements.

If $|\psi_i(\mathfrak{A})| = \alpha$ then B_i contains β elements.

Let $\mathfrak{B}$ be a model for L with the domain $\bigcup_{1 \leqq i \leqq 2^k} B_i$ and with $P_i(\mathfrak{B}) = \cup (B_j\colon$ The i-th conjunct in ψ_j is P_i, $j = 1, \cdots, 2^k)$. By the methods of the previous section, it is easy to show that the structures $\langle A, P_1(\mathfrak{A}), \cdots, P_{2k}(\mathfrak{A})\rangle$, $\langle B, P_1(\mathfrak{B}), \cdots, P_{2^k}(\mathfrak{B})\rangle$ are α, β, n-locally isomorphic for every n. Therefore, $\mathfrak{A}^{\alpha} \equiv {}^{\beta}\mathfrak{B}$.

THEOREM 3.2. *For every α, every integer $m > 0$ and every α-model $\mathfrak{A}$ of $T(Q)$, there exists $\mathfrak{B}$ such that $\mathfrak{A}^{\alpha} \equiv {}^{\aleph_0}_{m}\mathfrak{B}$.*

PROOF. The result follows in the same way as the proof of Theorem 3.1.

REMARK 3.1. It is impossible to improve Theorem 3.3 as it was impossible to improve Lemmas 2.2 and 2.4.

THEOREM 3.3. *$T(\alpha) = T(\aleph_0)$ for every α, and $T(\aleph_0)$ is decidable.*

PROOF. The result follows in the same way as the proof of Theorem 2.3.

REMARK 3.2. Instead of dealing with the α-interpretation for $L(Q)$, it is possible also to deal with the equi-cardinality interpretation, namely, for infinite models of first order language, the quantifier Q is interpreted as "there exist as many elements as in the whole model". Denote this interpretation by $\models_{eq}$ and let T be any first order theory. Define

$T(eq) = \{\phi\colon \phi$ is a sentence in $L(Q)$ and for every infinite model $\mathfrak{A}$ if $\mathfrak{A} \models T$ then $\mathfrak{A} \models_{eq} \phi\}$.

One can easily show that for every first order theory T, $T(eq) = \bigcap_{\alpha \geqq \aleph_0} T(\alpha)$. Hence, if $T(\alpha) = T(\aleph_0)$ for every α, we obtain $T(eq) = T(\aleph_0)$. It follows that if T is the first order theory of one equivalence relation or T is the first order theory of k unary relations then $T(eq)$ is decidable. The decidability of $T(eq)$, when T is the theory of k unary relations, was proved by Slomson [7] by completely different methods.

4. The theory of well-ordered sets and the theory of one unary function

The concept of α, β, n-local isomorphism is useful at least in two additional cases. The first one is the theory of well-ordered sets. Here it is used to prove the following theorem.

THEOREM 4.1. *Let $\alpha, \beta > \aleph_0$ and let n be any positive integer. Then for every well-ordered set $\mathfrak{A}$, $|\mathfrak{A}| \geqq \alpha$, there exists a well-ordered set $\mathfrak{B}$ such that $\mathfrak{A}^\alpha \equiv {}^\beta_n\mathfrak{B}$.*

Let L be a first order language for ordered sets. Denote by T_α the set of all sentences in $L(Q)$ that hold in every well-ordered set of power greater or equal to α. By Theorem 4.1, one can obtain immediately that $T_\alpha = T_\beta$ for every $\alpha, \beta, > \aleph_0$. The interested reader will find a comprehensive consideration of this subject in Slomson [8], where (using results of Lipner [5]) all the preparations needed for the proof of Theorem 4.1 are made.

The second case, where the concept of α, β, n-local isomorphism can be used, is the case when T is the theory of one unary function formulated in a language which contains only one binary predicate (in addition to logical constants). We mention it without a proof.

THEOREM 4.2. *For every regular $\alpha > \aleph_1$, for every $\aleph_1$-model $\mathfrak{B}$ of $T(Q)$ and for every integer $n > 0$, there exists $\mathfrak{A}$ such that $\mathfrak{A}^\alpha \equiv {}^{\aleph_1}_{n}\mathfrak{B}$.*

The proof of this theorem has many steps and tedious details and constitutes a separate paper. By the last theorem and by a theorem of Fuhrken [3], we obtain

THEOREM 4.3. *$T(\alpha) = T(\aleph_1)$ for every regular $\alpha > \aleph_1$.*

On the other hand it is easy to see that $T(\beta) \neq T(\aleph_1)$ for every singular β. In particular, if $P(x, y)$ is the binary predicate for the relation "y is the image of x by the given function", then the sentence $\neg\, Qy\exists xP(x, y) \rightarrow \exists yQxP(x, y)$ is in $T(\aleph_1)$ but not in $T(\beta)$ for every singular β.

Moreover, it is not difficult to show that $T(\aleph_1) \neq T(\aleph_0)$ since the sentence

$$\exists! y[[\forall xP(x, y) \rightarrow \neg\, \exists uP(u, x)] \wedge \forall z[z \neq y \wedge \neg\, P(z, y) \rightarrow \exists! wP(w, z)]]$$

$\rightarrow \forall y_1 Qx_1 \neg\, P(x_1, y_1)$ belongs to $T(\aleph_0)$ but not to $T(\aleph_1)$.

While $T(\aleph_0)$ is decidable by Rabin [6], it is unknown whether $T(\aleph_1)$ is decidable or not.

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